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Compatibility requirements for yield-line mechanisms

S.R. Denton *

Parsons Brinckerhoff, Bristol, UK

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Abstract

Yield-line analysis proves a powerful and convenient method for establishing the peak load capacity of reinforced concrete slabs. Its application requires compatible mechanisms to be postulated, comprising rigid regions intersecting at yield-lines where relative rotations may occur. A systematic procedure is described for checking the compatibility of postulated yield-line mechanisms. The similarity between this method and equilibrium requirements in a plane pin-jointed truss is highlighted, and an analogy between yield-line mechanism compatibility and statical determinacy in pin-jointed trusses is established.

Keywords: Plasticity; Limit analysis; Yield-line; Compatibility

1. Introduction

Yield-line analysis is an upper bound method of limit analysis which enables an estimate of the ultimate or peak load capacity of a slab to be derived, assuming a flexural mode of failure and perfect plasticity. The method is applied by postulating a compatible mechanism of displacement (or velocity) comprising rigid regions intersecting at yield-lines where relative rotations may occur. An estimate of the ultimate load then follows from the upper bound theorem of limit analysis by equating the rate of internal energy dissipation in the yield-lines to the rate of work done by the applied loading as the slab deforms in this mechanism. The method was originally pioneered by Johansen (1962) and is extensively treated by Park and Gamble (1980).

One problem with yield-line theory is that "mechanisms" are occasionally proposed which turn out to be geometrically incompatible and therefore illegitimate. In this paper, a systematic procedure for checking the compatibility of yield-line mechanisms is presented. In developing this procedure, it is convenient to recognise two compatibility requirements that must be satisfied by yield-line mechanisms. Firstly, the arrangement of yield-lines must be such that a pattern of displacements can occur through rotation of the

* Formally Cambridge University Engineering Department, University of Cambridge, UK. Address: Queen Victoria House, Redland Hill, Redland, Bristol, BS6 6US.

E-mail address: dentons@pbworld.com (S.R. Denton).

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yield-lines alone with the regions between the yield-lines remaining rigid; here this condition is termed the rotational compatibility requirement for a yield-line mechanism. Secondly, this pattern of displacements must be compatible with the boundary conditions of the problem, which are generally conditions on displacement or rotation at supports. For many yield-line mechanisms, the requirement for compatibility with boundary conditions can be seen to be satisfied by inspection.

A method is first presented for checking that the rotational compatibility requirement of a yield-line mechanism is satisfied. A comparison of this method and the requirements for equilibrium in pin-jointed trusses leads to an analogy between the compatibility of a yield-line mechanism and the existence of states of self-stress in a pin-jointed truss whose members lie along the hinge lines of the mechanism. This method is then generalised to address both the rotational compatibility requirement and the requirement for compatibility with boundary conditions. This general method is suitable for cases where boundary conditions cannot be seen to be satisfied by inspection and can be particularly relevant when slabs have free edges or internal supports.

Yield-line mechanisms in plane slabs only are considered and small deflections are assumed. Since the question of compatibility is quite independent of the strength actually developed at yield-lines, no distinction needs be drawn between yield-lines with strength and other lines of discontinuity in rotation between portions of the slab or surroundings (such as occur at simply supported edges with no flexural strength). Here, the term "yield-line" covers all such lines of discontinuity in rotation, and all points where yield-lines meet are termed "nodes". Of course, each rigid slab portion will have an axis of rotation, which may or may not coincide with one of the surrounding yield-lines.

2. Rotational compatibility requirement for yield-line mechanisms

Since yield-line mechanisms comprise rigid regions intersecting at yield-lines, between nodal points where the yield-lines themselves intersect, which must be straight and have constant rotation. The compatibility of a yield-line mechanism may then be checked by ensuring that the following conditions are satisfied at every nodal point:

$$\begin{aligned} &\sum_{i=1}^{n} \theta_i \sin\phi_i = 0, \\ &\sum_{i=1}^{n} \theta_i \cos\phi_i = 0, \end{aligned} \tag{1}$$

where n is the number of yield lines which meet at the node; θ_i, the rotations in the yield-lines, and ϕ_i, the anti-clockwise angles between some reference axis in the slab plane and the yield-lines. A consistent sign convention is required to account for the sense of the rotations in the yield-lines, i.e. whether they are hogging or sagging. In this paper, hogging is considered positive.

These criteria ensure that the vector sum of the yield-line rotations at each nodal point is zero. Such an approach is valid since small rotations are assumed. If a circular path is followed at a small radius around a nodal point then, as each yield-line is crossed, the difference in slope in any chosen direction between the rigid zones on either side of the yield-line will be equal to the component of the yield-line rotation in that direction, assuming small rotations. Clearly, if a *complete* circular path is followed then the total difference in slope must be zero and therefore the sum of the components of the rotations of all the yield-line which meet at the node, resolved in any chosen direction, must also equal zero. Eq. (1) ensures that this requirement is satisfied.

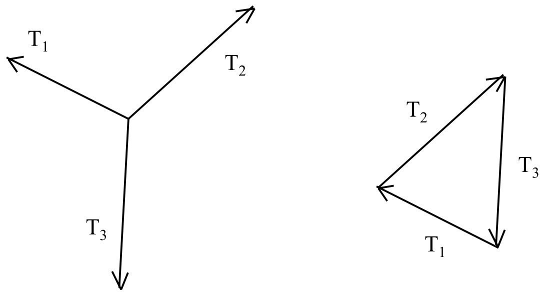

Fig. 1. Polygon of forces.

3. Equilibrium requirements for pin-jointed trusses or frames

In the analysis of two-dimensional pin-jointed trusses or frames (with straight members), equilibrium is achieved by ensuring that the polygon of forces throughout the truss is closed (Fig. 1). Assuming the truss members have no self-weight and that members are not loaded between joints, this condition leads to the requirements that, between pin-joints or nodes, bar forces must be constant and that, at every node, the following equations must be satisfied:

$$\sum_{i=1}^{n} T_i \sin\phi_i = 0, \qquad \sum_{i=1}^{n} T_i \cos\phi_i = 0, \tag{2}$$

where n is the number of bars which meet at the node; T_i, the bar tensions, and ϕ_i, the anti-clockwise angles between some reference axis in the truss plane and the bars.

By comparing Eqs. (1) and (2), it is clear that an analogy exists between the compatibility requirements for yield-line mechanisms and equilibrium requirements for pin-jointed trusses, with yield-line rotations in the former being replaced by bar tensions in the latter.

That 'static-geometric analogies' of this form exist has been known for some time. Calladine (1983) provides a more general exposition of which the particular analogy shown above is a special case, although it does not seem to have been highlighted before in this context. Furthermore, it is possible to extend this particular analogy to enable methods available for finding possible states of self-stress in pin-jointed trusses to be used to establish whether yield-line mechanisms satisfy the requirement for rotational compatibility.

4. Analogy between states of self-stress in pin-jointed trusses and the rotational compatibility requirement for yield-line mechanisms

Before exploring the analogy between states of self-stress in pin-jointed trusses and the compatibility of yield-line mechanisms, it is helpful to introduce the term "equivalent pin-jointed truss". Such a truss is developed from a yield-line mechanism by replacing the yield-lines with truss members and the intersection points between yield-lines with pin joints. The members in the equivalent pin-jointed truss have no self-weight.

For a yield-line mechanism to satisfy the requirement for rotational compatibility, it is necessary and sufficient for there to be a system of rotations constant along each yield-line which satisfy Eq. (1) at every intersection point. Recognising the analogy between the rotational compatibility requirement for yield-line mechanisms and the equilibrium requirements for pin-jointed trusses, it follows that for a yield-line

Table 1
Analogy between pin-jointed trusses and yield-line mechanisms

Equivalent pin-jointed truss	Yield-line mechanism
No state of self-stress	Incompatible
Single state of self-stress	Compatible with single degree of freedom
n States of self-stress	Compatible with n degrees of freedom

mechanism to be compatible its equivalent pin-jointed truss must be capable of sustaining at least one self-equilibrating system of bar forces (with no external load). Furthermore, it may be seen that the number of degrees of freedom of a compatible yield-line mechanism is equal to the number of independent states of self-stress in its equivalent pin-jointed truss. Thus, for a yield-line mechanism to have a single degree of freedom, its equivalent pin-jointed truss must have a single state of self-stress.

The number of degree of freedom of a yield-line mechanism is relevant since, from the upper bound theorem, it follows that for plane slabs, compatible yield-line mechanisms with more than one degree of freedom can never generate lower critical collapse loads than yield-line mechanisms with a single degree of freedom (i.e. mechanisms where, if the rotation in a single yield-line is specified, the rotation of all other yield-lines follows uniquely). In general, therefore, only mechanisms with a single degree of freedom are of interest when the load capacity of a plane slab is sought.

The analogy between the rotational compatibility of yield-line mechanisms and states of self-stress in their equivalent pin-jointed trusses is summarised in Table 1.

5. States of self-stress in pin-jointed trusses

A systematic method for analysing indeterminate pin-jointed trusses is presented by Pellegrino and Calladine (1986). Here, however, we are concerned only with a part of their more general solution since we seek solely the states of self-stress which may be sustained by our equivalent pin-jointed truss; the number of independent states of self-stress being equal to the number of degrees of freedom of the corresponding yield-line mechanism and the tensions in each member corresponding to the yield-line rotations.

Suppose that an equivalent pin-jointed truss has b members and j joints, then, with reference to Fig. 2, the equilibrium conditions at each joint can be written in matrix form as

$$\mathbf{A} \cdot \mathbf{t} = 0, \tag{3}$$

where the equilibrium matrix, $\mathbf{A}$, and the bar tensions vector, $\mathbf{t}$, are given by

$$\mathbf{A} = \underbrace{\begin{bmatrix} \cdot & \overset{(l)}{\cdot} & \cdot & \overset{(m)}{\cdot} & \cdot & \overset{(n)}{\cdot} & \cdot \\ \cdot & \frac{(x_i - x_h)}{\sqrt{(x_i - x_h)^2 + (y_i - y_h)^2}} & \cdot & \frac{(x_i - x_j)}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}} & \cdot & \frac{(x_i - x_k)}{\sqrt{(x_i - x_k)^2 + (y_i - y_k)^2}} & \cdot \\ \cdot & \frac{(y_i - y_h)}{\sqrt{(x_i - x_h)^2 + (y_i - y_h)^2}} & \cdot & \frac{(y_i - y_j)}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}} & \cdot & \frac{(y_i - y_k)}{\sqrt{(x_i - x_k)^2 + (y_i - y_k)^2}} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}}_{b} \left.\vphantom{\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}}\right\} 2j$$

$$\mathbf{t} = [\, t_1 \quad \cdots \quad t_l \quad \cdots \quad t_m \quad \cdots \quad t_n \quad \cdots \quad t_b \,]^{\mathrm{T}}.$$

and t_n is the tension in bar n.

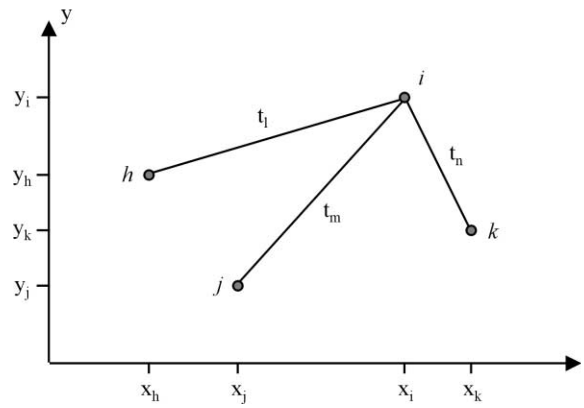

Fig. 2. Members connected at joint *i* in a pin-jointed truss.

In the equilibrium matrix, **A**, each column corresponds to a particular bar and two rows relate to each joint since each joint generates two equilibrium equations. When a bar is not connected to a particular joint then the rows relating to that joint will have a zero in the column corresponding to that bar.

The values of **t** which satisfy Eq. (3) correspond to the possible states of self-stress in the truss. Identifying these solutions for **t** is a problem of linear algebra for which simple solution procedures are available (Strang, 1988); the values of **t** which satisfy Eq. (3) span the so-called nullspace of **A** and the number of independent states of self-stress which can be sustained by the structure is given by br_A, where r_A is the rank of the equilibrium matrix, **A**.

6. Examples

Five yield-line patterns are shown in Fig. 3 from which the relationship between their compatibility and states of self-stress in the equivalent pin-jointed trusses may be observed. In each case, the outer yield-lines connect to a rigid surrounding slab satisfying any boundary conditions. The number of members in the equivalent pin-jointed trusses, rank of the equilibrium matrices, number of independent states of self-stress and corresponding yield-line mechanism compatibilities are summarised in Table 2.

By considering the effect of increasing the length of any of the members in the equivalent truss corresponding yield-line pattern 1, it is clear that no system of self-stress may be sustained by the structure and the corresponding yield-line mechanism is not compatible. The addition of a further yield-line, as in pattern 2, leads to a single redundancy in the equivalent pin-jointed truss and a compatible yield-line mechanism.

Whilst patterns 3 and 4 comprise the same number of yield-lines and intersections, the former is not compatible whilst the latter is. The difference arises because yield-line *cf* is parallel to *bd* and *ae* in pattern 4 whereas this is not the case in pattern 3. Thus, pattern 4 has one fewer independent "equilibrium" equations than pattern 3 and the rank of its equilibrium matrix is reduced by one. The present approach can enable such special cases to be found analytically; one method for achieving this is described in the following section.

Pattern 5 clearly has two degrees of freedom, namely deflection of the central point with no rotation of the central square of yield-lines; and no deflection of the central point but rotation of different sense in the inner and outer diagonals. Its equivalent pin-jointed truss has two independent states of self-stress.

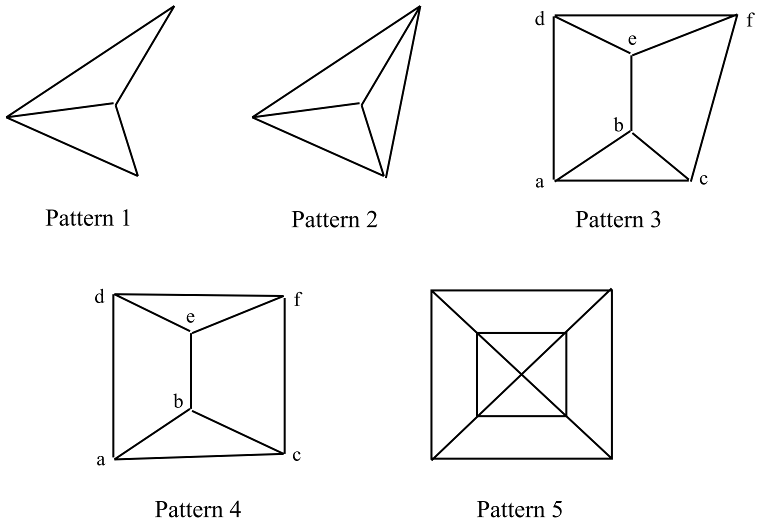

Fig. 3. Examples.

Table 2
Examples

Yield-line pattern (see Fig. 3)	Equivalent pin-jointed truss			Yield-line mechanism	
	Number of members	Rank of equilibrium matrix	Number of states of self-stress	Compatible	Number of degrees of freedom
1	5	5	0	No	–
2	6	5	1	Yes	1
3	9	9	0	No	–
4	9	8	1	Yes	1
5	16	14	2	Yes	2

7. Finding compatible yield-line mechanisms

Some arrangements of yield-lines satisfy the requirement for rotational compatibility irrespective of the length of the individual yield-lines, e.g. pattern 2 in Fig. 3. However, for other topologies, only certain special cases are compatible; whilst patterns 3 and 4 in Fig. 3 have the same topology, or arrangement of yield-lines and joints, pattern 4 is compatible whereas pattern 3 is not. Such special cases can be found analytically by examining the rank of the equilibrium matrix of the equivalent pin-jointed truss as illustrated by the following example. Alternatively, the same problem could be formulated as a generalised eigenvector problem.

Consider the yield-line pattern shown in Fig. 4. The outer yield-lines connect to a rigid surrounding slab satisfying any boundary conditions. The co-ordinates of intersections a,b,c,d, and e and the y ordinate of f are fixed, with values as shown. A value for the x ordinate of f, denoted by x, is sought such that the yield-line mechanism is compatible and that the area of each of the rigid regions bounded by the yield-lines is

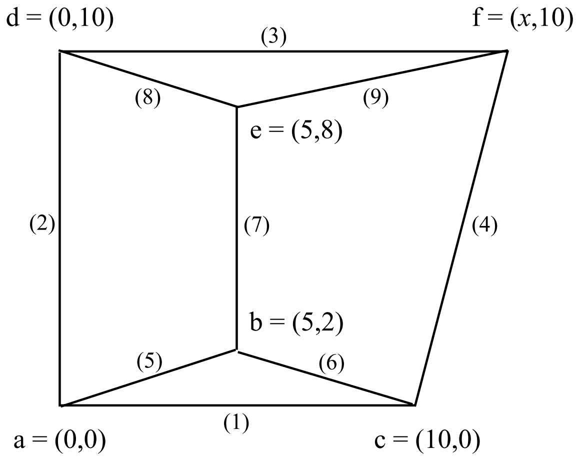

Fig. 4. Yield-line pattern with x ordinate of node f unspecified.

greater than zero. The latter condition is used to eliminate states of self-stress in co-linear truss members alone since these have no relevance in yield-line analysis.

The rank of the equilibrium matrix is unaffected by multiplying all the terms in one row or column by a constant factor or by adding all the terms from one row to another. An investigation of the rank of the equilibrium matrix **A**, given in Eq. (3), may therefore be simplified since the denominator of each term may be cancelled to give a new matrix with the same rank as **A**. For the geometry shown in Fig. 4, this matrix has nine columns, each corresponding to one of the nine members, and twelve rows since each of the six joints generates two equilibrium equations. The order in which the joints are taken is, of course, arbitrary; using the order a,c,b,e,d,f, this matrix is given by

$$
\begin{array}{ccccccccc}
(1) & (2) & (3) & (4) & (5) & (6) & (7) & (8) & (9)
\end{array}
$$

$$
\begin{bmatrix}
-10 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 \\
0 & -10 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
10 & 0 & 0 & 10-x & 0 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & -10 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & -5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 & -6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 5-x \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & -2 & -2 \\
0 & 0 & -x & 0 & 0 & 0 & 0 & -5 & 0 \\
0 & 10 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & x & x-10 & 0 & 0 & 0 & 0 & x-5 \\
0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 & 2
\end{bmatrix}.
$$

The matrix may be simplified further without altering its rank using Gaussian elimination (Strang, 1988). With some exchanging of rows, this leads to

$$\begin{bmatrix} -10 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & x-10 & 0 & 0 & 0 & 0 & x-5 \\ 0 & 0 & 0 & -10 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 5-x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10-x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, for the majority of values of x, the equilibrium matrix is of full rank and therefore the corresponding yield-line pattern is not compatible. Two special cases appear to be of interest however, namely $x = 0$ and 10. In the former case, the co-ordinates of joint f are identical to d, thus the area of the rigid region def equals zero, violating one of the requirements for an acceptable solution in this example. In the latter case no such complications arise; the rank of the equilibrium matrix is one less than the number of columns so this geometry corresponds to a compatible yield-line pattern with a single degree of freedom.

8. Compatibility with boundary conditions

For some problems, it is not immediately clear whether postulated yield-line mechanisms satisfy the requirement for compatibility with boundary conditions. This can be particularly relevant to problems involving slabs with free edges or internal supports.

When slabs have free edges, it is frequently necessary to consider collapse mechanisms with yield-lines that intersect these free edges. Such an example is illustrated in Fig. 5a. Whilst the rotational compatibility criterion set out in Eq. (1) is a necessary condition for mechanisms of this type, it is not a sufficient condition for compatibility. Furthermore, it is not immediately clear how the equivalent pin-jointed truss should be constructed for such a case nor how boundary conditions (e.g. constraints at supports) should be taken into account.

The mechanism will be compatible provided the yield-lines can be shown to form part of any equivalent truss which extends beyond the slab boundaries but has members within the slab boundaries solely at the yield-line locations and which has a state of self-stress giving rise to tensions in the members corresponding to these yield-lines. In addition, constraints at the supports must not be violated by the resulting mechanism. An equivalent pin-jointed truss that satisfies these requirements is illustrated in Fig. 5b.

A general approach is available, however, that is far more amenable to numerical analysis and which enables the requirements for rotational compatibility and compatibility with the boundary conditions to be considered together. The compatibility criterion set out in Eq. (1) is applied at any nodal points within the slab boundaries, but no constraint is placed on the rotation of yield-lines where they intersect free edges. Additional equations are then introduced to enforce the support constraints directly.

The yield-line mechanism shown in Fig. 5c will be used to illustrate this approach. The co-ordinates of the intersections of the yield-lines and free edges of the slab, namely b, d and f, are fixed and the co-ordinates of node i, denoted by (x_i, y_i), are sought such that the resulting yield-line mechanism is compatible and node i lies *within* the slab boundaries.

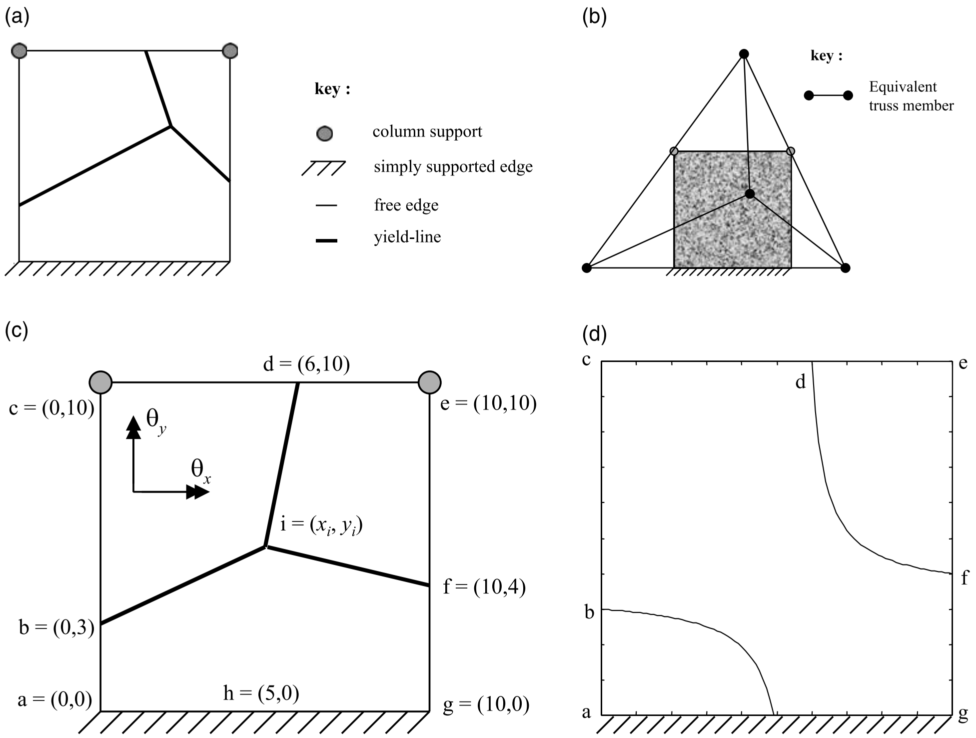

Fig. 5. (a) Yield-line mechanism with yield-lines intersecting free edges, (b) equivalent pin-jointed truss with members extending beyond slab boundaries, (c) yield-line mechanism with yield-lines intersecting free edges and position of node *i* unspecified, and (d) locus of positions of node *i* to give compatible mechanisms.

Whilst it is possible to extend the static-geometric analogy, so that support constraints are considered as equilibrium requirements in some fashion, such an approach becomes rather contrived and since it does not serve to clarify the problem it is not pursued here. Instead, the actual rotations of the various rigid regions and their displacements are used. For consistency, the yield-line rotations and their associated compatibility criteria are therefore referred to as such rather than, by analogy, as "tensions" and "equilibrium" equations. Similarly, the "equilibrium" matrix, **A**, is replaced by the compatibility matrix, **C**, which now has extra rows and columns to enforce the boundary conditions.

The slab supports place constraints on the rotations of the adjoining rigid-regions and their displacement. A column support solely fixes the displacement at its location. A simple support fixes the displacement at a point on the support and the axis of rotation of the adjoining rigid region. For a fixed support the displacement and the rotation of the adjoining rigid region are fixed.

A single rigid region is chosen, in the present example the region *bcdi*, and assigned unknown rotational components about the x and y axis denoted by θ_x and θ_y, as illustrated in Fig. 5c. In addition, the displacement of a point chosen within this region is denoted by δ; in this example it is convenient to choose the column support, c, so δ has a value of zero. It is now straightforward to determine the rotations of all other rigid regions as linear functions of θ_x, θ_y and the yield-line rotations since all rotations are assumed to be small and may therefore be treated as vectors. Similarly, the displacement at any point in the slab can be determined. The conditions that must be satisfied at the support locations may therefore be written as a set of linear equations of the yield-line rotations, θ_x, θ_y and δ.

It is more convenient to use rotation coefficients instead of the yield-line rotations proper in these equations, where the rotation coefficient for yield-line n, denoted by ψ_n, is defined as

$$\psi_n = \frac{\theta_n}{L_n}, \tag{4}$$

where L_n is the length of yield-line and θ_n its rotation. In the present example, the rotation coefficients of the yield-lines *id*, *if* and *ib* are denoted ψ_1, ψ_2 and ψ_3, respectively.

The set of linear equations which enforce the boundary conditions are developed by considering each of the supports in turn. The component of the rotation of region *abifgh* about the y-axis must be zero since it adjoins a simple-support which lies parallel to the x-axis. The component of rotation of this region about the y-axis is given by the sum of the rotation of region *bcdi* about the y-axis, θ_y, and the component of the rotation of yield-line *ib* about the y-axis, thus,

$$\theta_y - \psi_3(y_b - y_i) = 0. \tag{5a}$$

The displacement at the column support e and also at some point along the simple support, say h, must also be equal to zero, thus,

$$\delta + \theta_y(x_i - x_c) - \theta_x(y_i - y_c) + \lfloor\theta_y - \psi_1(y_i - y_d)\rfloor(x_e - x_i) - [\theta_x - \psi_1(x_i - x_d)](y_e - y_i) = 0, \tag{5b}$$

$$\delta + \theta_y(x_i - x_c) - \theta_x(y_i - y_c) + \lfloor\theta_y - \psi_3(y_b - y_i)\rfloor(x_h - x_i) - [\theta_x - \psi_3(x_b - x_i)](y_h - y_i) = 0. \tag{5c}$$

As noted previously, the requirement that the displacement at support c is zero is enforced by

$$\delta = 0. \tag{5d}$$

The rotational compatibility criteria from Eq. (1), applied to node i, and the support condition criteria given in Eqs. (5a)–(5d) may be combined in matrix form as

$$\mathbf{C} \cdot \boldsymbol{\theta} = 0. \tag{6}$$

Where in the present example, the compatibility matrix, $\mathbf{C}$, and the vector $\boldsymbol{\theta}$ are given by

$$\mathbf{C} = \begin{bmatrix} x_i - 6 & x_i - 10 & x_i & 0 & 0 & 0 \\ y_i - 10 & y_i - 4 & y_i - 3 & 0 & 0 & 0 \\ 0 & 0 & y_i - 3 & 0 & 1 & 0 \\ 40 - 4y_i & 0 & 0 & 0 & 10 & 0 \\ 0 & 0 & 5y_i + 3x_i - 15 & 10 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\boldsymbol{\theta} = [\psi_1 \quad \psi_2 \quad \psi_3 \quad \theta_x \quad \theta_y \quad \delta]^{\mathrm{T}}.$$

The first two rows of the compatibility matrix, $\mathbf{C}$, enforce Eq. (1) at node i. The remaining four rows enforce the support condition criteria, namely Eqs. (5a)–(5d).

Eq. (6) has the same form as Eq. (3) and may be solved in a similar manner. Conveniently, the nullspace of $\mathbf{C}$ contains both the compatible yield-line rotation coefficients and the rotation of one of the rigid-regions and its deflection at one point from which all other rotations and deflections follow.

Here, however, we seek solely to identify the co-ordinates of i such that the compatibility matrix is rank deficient. This may conveniently be achieved by performing Gaussian elimination on the compatibility matrix and then identifying cases where the product of the terms on its leading diagonal is equal to zero. Such an approach leads to the condition that for the yield-line mechanism to be compatible

$$y_i = \frac{55x_i - 270}{16x_i - 90}. \tag{7}$$

The locus of points satisfying this relationship and lying within the slab boundaries is shown in Fig. 5d. It is of interest to note that two families of compatible mechanisms are identified; when *i* lies between *d* and *f* all yield-lines have the same sense (i.e. all hogging or sagging), whereas when *i* lies between *b* and the simply supported edge this is not the case.

When slabs are continuous over supports, yield-line mechanisms can be postulated in which supports lie within the area bounded by the outermost yield-lines. The approach described above may readily be adapted to such cases to enforce the conditions required at such internal supports.

9. Conclusions

A systematic method is presented for checking the compatibility of yield-line mechanisms and an analogy between compatibility of yield-line mechanisms and states of self-stress in pin-jointed trusses is highlighted. Applying the analogy, it becomes possible to use techniques developed to investigate states of self-stress in pin-jointed trusses to determine whether yield-line mechanisms satisfy the requirement for rotational compatibility. Furthermore, the approach allows the number of degrees of freedom of yield-line mechanisms to be readily established. A development of the approach to account directly for boundary conditions (or support constraints) has been presented.

The present approach should enable computational methods developed for problems of linear algebra, and in particular those for analysing two dimensional pin-jointed frameworks, to be applied directly to problems of yield-line mechanism compatibility.

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